Chapter 10

Infinite Series

Section 10.1 Power Series (pp. 477–487)

Exploration 1 Finding Power Series for Other Functions

1. **(a)** $\frac{1}{1+x}=\frac{1}{1-(-x)}$

$=1+(-x)+(-x)^2+(-x)^3+\cdots+(-x)^n+\cdots$

$=1-x+x^2-x^3+\cdots+(-1)^n x^n+\cdots$

This series converges for $-1<-x<1$, which is equivalent to $-1<x<1$. The interval of convergence is $(-1, 1)$.

(b) $\frac{x}{1+x}=\frac{x}{1-(-x)}$

$=x-x^2+x^3-x^4+\cdots+(-1)^n x^{n+1}+\cdots$

The interval of convergence is $(-1, 1)$.

(c) $\frac{1}{1-2x}=1+(2x)+(2x)^2+(2x)^3+\cdots+(2x)^n+\cdots$

$=1+2x+4x^2+8x^3+\cdots+2^n x^n+\cdots$

This series converges for $-1<2x<1$, which is equivalent to $-\frac{1}{2}<x<\frac{1}{2}$. The interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

(d) $\frac{1}{x}=\frac{1}{1+(x-1)}$

$=1-(x-1)+(x-1)^2-(x-1)^3+\cdots+(-1)^n(x-1)^n+\cdots$

This series converges for $-1<x-1<1$, which is equivalent to $0<x<2$. The interval of convergence is $(0, 2)$.

2. $\frac{1}{3x}=\frac{1}{3}\cdot\frac{1}{x}$

$=\frac{\frac{1}{3}}{1+(x-1)}$

$=\frac{1}{3}-\frac{1}{3}(x-1)+\frac{1}{3}(x-1)^2-\frac{1}{3}(x-1)^3+\cdots+\frac{(-1)^n}{3}(x-1)^n+\cdots$

As in Problem 1 part (d), the interval of convergence is $(0, 2)$.

Exploration 2 Finding a Power Series for $\tan^{-1} x$

1. $\dfrac{1}{1+x^2} = 1 - x^2 + (x^2)^2 - (x^2)^3 + \cdots + (-1)^n (x^2)^n + \cdots$

$= 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$

The series converges for $-1 < x^2 < 1$, which is equivalent to $-1 < x < 1$. The interval of convergence is $(-1, 1)$.

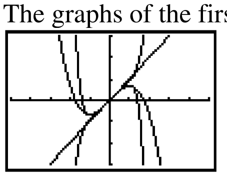
2. $\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt$

$$= \int_0^x (1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \cdots)\,dt$$

$$= \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots + (-1)^n \frac{t^{2n+1}}{2n+1} + \cdots \right]_0^x$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} \cdots.$$

3. The graphs of the first four partial sums appear to be converging on the interval $(-1, 1)$.



[–5, 5] by [–3, 3]

4. When $x = 1$, the series becomes $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$.

This series does appear to converge. The terms are getting smaller, and because they alternate in sign they cause the partial sums to oscillate above and below a limit. The two calculator statements shown below will cause the successive partial sums to appear on the calculator each time the ENTER button is pushed. The partial sums will appear to be approaching a limit of $\pi/4$ (which is $\tan^{-1}(1)$), although very slowly.

```
0→N:1→T
                1
N+1→N:T+(-1)^N/(
2N+1)
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Exploration 3 A Series with a Curious Property

1. Recall $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$.

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$f'(x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots + \frac{nx^{n-1}}{n!} + \cdots$$

$$= 1 + \frac{2x}{2\cdot 1} + \frac{3x^2}{3\cdot 2\cdot 1} + \cdots + \frac{nx^{n-1}}{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} + \cdots$$

$$= f(x)$$

2. $f(0) = 1 + 0 + 0 + \cdots = 1.$

3. Since this function is its own derivative and takes on the value 1 at $x = 0$, we suspect that it must be e^x.

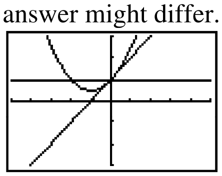
4. If $y = f(x)$, then $\frac{dy}{dx} = y$ and $y = 1$ when $x = 0$.

5. The differential equation is separable.

$$\frac{dy}{y} = dx$$
$$\int \frac{dy}{y} = \int dx$$
$$\ln|y| = x + C$$
$$y = Ke^x$$
$$1 = Ke^0 \Rightarrow K = 1$$

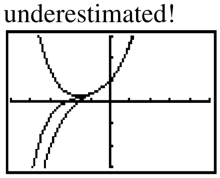
Therefore, $y = e^x$.

6. The first three partial sums are shown in the graph below.
It is risky to draw any conclusions about the interval of convergence from just three partial sums, but so far the convergence to the graph of $y = e^x$ only looks good on $(-1, 1)$. Your answer might differ.



$[-5, 5]$ by $[-3, 3]$

7. The next three partial sums show that the convergence extends outside the interval $(-1, 1)$ in both directions, so $(-1, 1)$ was apparently an underestimate. Your answer in #6 might have been better, but unless you guessed "all real numbers" you still underestimated!



$[-5, 5]$ by $[-3, 3]$

Quick Review 10.1

1. $u_1 = \frac{4}{1+2} = \frac{4}{3}$

$u_2 = \frac{4}{2+2} = \frac{4}{4} = 1$

$u_3 = \frac{4}{3+2} = \frac{4}{5}$

$u_4 = \frac{4}{4+2} = \frac{4}{6} = \frac{2}{3}$

$u_{30} = \frac{4}{30+2} = \frac{4}{32} = \frac{1}{8}$

2. $u_1 = \frac{(-1)^1}{1} = -1$

$u_2 = \frac{(-1)^2}{2} = \frac{1}{2}$

$u_3 = \frac{(-1)^3}{3} = -\frac{1}{3}$

$u_4 = \frac{(-1)^4}{4} = \frac{1}{4}$

$u_{30} = \frac{(-1)^{30}}{30} = \frac{1}{30}$

3. **(a)** Since $\frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$, the common ratio is 3.

(b) $2(3^9) = 39{,}366$

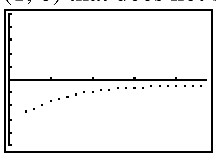
(c) $a_n = 2(3^{n-1})$

4. **(a)** Since $\frac{-4}{8} = \frac{2}{-4} = \frac{-1}{2} = -\frac{1}{2}$, the common ratio is $-\frac{1}{2}$.

(b) $8\left(-\frac{1}{2}\right)^9 = -\frac{1}{64}$

(c) $a_n = 8\left(-\frac{1}{2}\right)^{n-1} = 8(-0.5)^{n-1}$

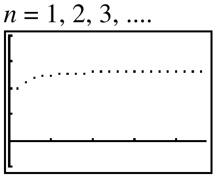
5. **(a)** We graph the points $\left(n, \dfrac{1-n}{n^2}\right)$ for $n = 1$, 2, 3, …. (Note that there is a point at (1, 0) that does not show in the graph.)



[0, 25] by [–0.5, 0.5]

(b) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1-n}{n^2} = 0$

6. **(a)** We graph the points $\left(n, \left(1+\dfrac{1}{n}\right)^n\right)$ for $n = 1, 2, 3, ...$

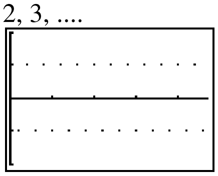


[0, 23.5] by [–1, 4]

(b) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n = e$

(See Section 9.2, Example 8)

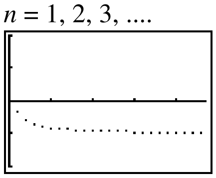
7. **(a)** We graph the points $(n, (-1)^n)$ for $n = 1$, 2, 3, ...



[0, 23.5] by [–2, 2]

(b) $\lim\limits_{n\to\infty} a_n$ does not exist because the values of a_n oscillate between –1 and 1.

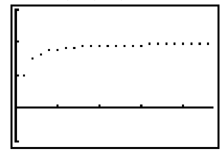
8. **(a)** We graph the points $\left(n, \dfrac{1-2n}{1+2n}\right)$ for $n = 1, 2, 3, ...$



[0, 23.5] by [–2, 2]

(b) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1-2n}{1+2n} = \lim\limits_{n\to\infty} \dfrac{-2}{2} = -1$

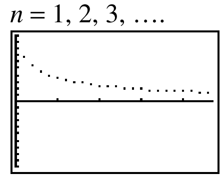
9. **(a)** We graph the points $\left(n,\ 2-\dfrac{1}{n}\right)$ for $n = 1, 2, 3, ...$



[0, 23.5] by [–1, 3]

(b) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(2-\dfrac{1}{n}\right) = 2$

10. **(a)** We graph the points $\left(n, \dfrac{\ln(n+1)}{n}\right)$ for $n = 1, 2, 3, ….$



[0, 23.5] by [–1, 1]

(b) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{\ln(n+1)}{n} = \lim\limits_{n\to\infty} \dfrac{\frac{1}{(n+1)}}{1} = 0$

Section 10.1 Exercises

1. **(a)** Let u_n represent the value of $*$ in the nth-term, starting with $n = 1$. Then $\dfrac{1}{u_1} = 1,\ -\dfrac{1}{u_2} = -\dfrac{1}{4},\ \dfrac{1}{u_3} = \dfrac{1}{9},$ and $-\dfrac{1}{u_4} = -\dfrac{1}{16},$ so $u_1 = 1,\ u_2 = 4,$ $u_3 = 9,$ and $u_4 = 16.$ We may write $u_n = n^2$, or $* = n^2$.

(b) Let u_n represent the value of $*$ in the nth-term, starting with $n = 0$. Then $\dfrac{1}{u_0} = 1,\ -\dfrac{1}{u_1} = -\dfrac{1}{4},\ \dfrac{1}{u_2} = \dfrac{1}{9},$ and $-\dfrac{1}{u_3} = -\dfrac{1}{16}$, so $u_0 = 1,\ u_1 = 4,\ u_2 = 9,$ and $u_3 = 16.$ We may write $u_n = (n+1)^2$, or $* = (n+1)^2$.

(c) If $*=3$, the series is $(-1)^3\left(\frac{-1}{1^2}\right)+(-1)^4\left(\frac{-1}{2^2}\right)+(-1)^5\left(\frac{-1}{3^2}\right)+(-1)^6\left(\frac{-1}{4^2}\right)+\cdots=1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots,$ which is the same as the desired series. Thus let $*=3$.

2. (a) Note that $a_0=1,\ a_1=\frac{1}{3},\ a_2=\frac{1}{9},$ and so on. Thus $a_n=\left(\frac{1}{3}\right)^n.$

(b) Note that $a_1=1,\ a_2=-\frac{1}{2},\ a_3=\frac{1}{3},$ and so on. Thus $a_n=\frac{(-1)^{n-1}}{n}.$

(c) Note that $a_0=5,\ a_1=0.5,\ a_2=0.05,$ and so on. Thus $a_n=5(0.1)^n=\frac{5}{10^n}.$

3. Different, since the terms of $\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n-1}$ alternate between positive and negative, while the terms of $\sum_{n=1}^{\infty}-\left(\frac{1}{2}\right)^{n-1}$ are all negative.

4. The same, since both series can be represented as $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots.$

5. The same, since both series can be represented as $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots.$

6. Different, since $\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n-1}=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$ but $\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{n-1}}=-1+\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\cdots.$

7.

n	term	Partial sum
1	$a_1=1$	$S_1=1$
2	$a_2=1.1$	$S_2=1+1.1=2.1$
3	$a_3=1.11$	$S_3=2.1+1.11=3.21$
4	$a_4=1.111$	$S_4=3.21+1.111=4.321$

We see that
$$\begin{aligned} S_1&>1\\ S_2&>2\\ S_3&>3\\ &\vdots\\ S_n&>n \end{aligned}$$
So $\lim_{n\to\infty}S_n\ge\lim_{n\to\infty}n=\infty$

The series diverges.

8.

n	term	Partial sum
1	$a_1 = 2$	$S_1 = 2$
2	$a_2 = -1$	$S_2 = 2-1=1$
3	$a_3 = 1$	$S_3 = 1+1=2$
4	$a_4 = -1$	$S_4 = 2-1=1$
5	$a_5 = 1$	$S_5 = 1+1=2$

We see that the sequence of partial sums is 2, 1, 2, 1, 2, 1 · · ·, which has no limit. Since the sequence of partial sums has no limit, the series diverges.

9.

n	term	Partial sum
1	$a_1 = \frac{1}{2}$	$S_1 = \frac{1}{2} = 1 - \frac{1}{2}$
2	$a_2 = \frac{1}{4}$	$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4}$
3	$a_3 = \frac{1}{8}$	$S_3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8} = 1 - \frac{1}{8}$

We see that $S_1 = 1 - \left(\frac{1}{2}\right)^1$

$$S_2 = 1 - \left(\frac{1}{2}\right)^2$$

$$S_3 = 1 - \left(\frac{1}{2}\right)^3$$

$$\vdots$$

$$S_k = 1 - \left(\frac{1}{2}\right)^k$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[1 - \left(\frac{1}{2}\right)^n\right] = 1 - 0 = 1$$

Since the sequence of partial sums converges the series converges. (The sum of the series is 1.)

10. The partial sums are 3, 3.5, 3.55, 3.555, etc. The limit of the partial sums is $3.\overline{5}$, or $3\frac{5}{9}$. The series converges.

11. Geometric series with $r = \frac{2}{3}$, $a = \left(\frac{2}{3}\right)^0 = 1$.

Converges: $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = 3$

12. Diverges, because the terms do not approach zero.

13. Geometric series with $r = \frac{2}{3}$, $a = \frac{5}{4}\left(\frac{2}{3}\right)^0 = \frac{5}{4}$.

Converges: $\sum_{n=0}^{\infty}\left(\frac{5}{4}\right)\left(\frac{2}{3}\right)^n = \frac{\frac{5}{4}}{1-\frac{2}{3}} = \frac{15}{4}$

14. Geometric series with $r = \frac{5}{4} > 1$. Diverges.

15. Diverges, because the terms alternate between 1 and –1 and do not approach zero.

16. Geometric series with $r = -0.1$ and $a = 3(-0.1)^0 = 3$.

Converges:

$$\sum_{n=0}^{\infty} 3(-0.1)^n = \frac{3}{1-(-0.1)} = \frac{3}{1.1} = \frac{3}{\frac{11}{10}} = \frac{30}{11}.$$

17. Geometric series with $r = -\frac{1}{\sqrt{2}}$ and $a = 1$.

Converges, since $|r| = \frac{1}{\sqrt{2}} = 0.707 < 1$:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sin^n\left(\frac{\pi}{4} + n\pi\right) \\
&= 1 + \left(-\frac{1}{\sqrt{2}}\right)^1 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^3 + \cdots \\
&= \sum_{n=0}^{\infty}\left(-\frac{1}{\sqrt{2}}\right)^n \\
&= \frac{1}{1-\left(-\frac{1}{\sqrt{2}}\right)} \\
&= \frac{\sqrt{2}}{\sqrt{2}+1} \\
&= \frac{\sqrt{2}\left(\sqrt{2}-1\right)}{\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)} \\
&= \frac{2-\sqrt{2}}{2-1} \\
&= 2-\sqrt{2}
\end{aligned}$$

18. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

Since the sequence of terms does not approach 0, the series diverges.

19. Geometric series with $r = \frac{e}{\pi} < 1$ (since $e < \pi$), and $a = \left(\frac{e}{\pi}\right)^1$.

Converges: $\sum_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^n = \frac{\frac{e}{\pi}}{1-\frac{e}{\pi}} = \frac{e}{\pi - e}$.

20. Geometric series with $r = \frac{5}{6}$ and $a = \frac{1}{6}\left(\frac{5}{6}\right)^0 = \frac{1}{6}$.

Converges: $\sum_{n=0}^{\infty}\frac{5^n}{6^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{6}\left(\frac{5}{6}\right)^n = \frac{\frac{1}{6}}{1-\frac{5}{6}} = 1$.

21. Since $\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty}(2x)^n$, the series converges when $|2x| < 1$ and the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Since the sum of the series is $\frac{1}{1-2x}$, the series represents the function $f(x) = \frac{1}{1-2x}$, $-\frac{1}{2} < x < \frac{1}{2}$.

22. Since $\sum_{n=0}^{\infty}(-1)^n(x+1)^n = \sum_{n=0}^{\infty}[-(x+1)]^n$, the series converges when $|-(x+1)| < 1$ and the interval of convergence is (–2, 0). Since the sum of the series is $\frac{1}{1-[-(x+1)]} = \frac{1}{x+2}$, the series represents the function $f(x) = \frac{1}{x+2}$, $-2 < x < 0$.

23. Since $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n (x-3)^n = \sum_{n=0}^{\infty}\left(\frac{3-x}{2}\right)^n$, the series converges when $\left|\frac{3-x}{2}\right| < 1$ and the interval of convergence is (1, 5). Since the sum of the series is $\frac{1}{1-\frac{(3-x)}{2}} = \frac{2}{x-1}$, the series represents the function $f(x) = \frac{2}{x-1}$, $1 < x < 5$.

24. For $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$, the series converges when $\left|\frac{x-1}{2}\right| < 1$ and the interval of convergence is (–1, 3). Since the sum of the series is $\frac{3}{1-\frac{(x-1)}{2}} = \frac{6}{3-x}$, the series represents the function $f(x) = \frac{6}{3-x}$, $-1 < x < 3$.

25. Since $\sum_{n=0}^{\infty}\sin^n x = \sum_{n=0}^{\infty}(\sin x)^n$, the series converges when $|\sin x| < 1$. Thus, the series converges for all values of x except odd integer multiples of $\frac{\pi}{2}$, that is, $x \neq (2k+1)\frac{\pi}{2}$ for integers k. Since the sum of the series is $\frac{1}{1-\sin x}$, the series represents the function $f(x) = \frac{1}{1-\sin x}$, $x \neq (2k+1)\frac{\pi}{2}$.

26. Since $\sum_{n=0}^{\infty}\tan^n x = \sum_{n=0}^{\infty}(\tan x)^n$, the series converges when $|\tan x| < 1$. Thus, the series converges for $-\frac{\pi}{4} + k\pi < x < \frac{\pi}{4} + k\pi$, where k is an integer. Since the sum of the series is $\frac{1}{1-\tan x}$, the series represents the function $f(x) = \frac{1}{1-\tan x}$, $-\frac{\pi}{4} + k\pi < x < \frac{\pi}{4} + k\pi$.

27.
$$f(x)=\frac{1}{1-2x}=\sum_{n=0}^{\infty}2^n x^n,\ -\frac{1}{2}<x<\frac{1}{2}$$
$$\frac{d}{x}\left(\frac{1}{1-2x}\right)=\frac{d}{dx}\left(1+2x+4x^2+8x^3+\cdots+2^n x^n+\cdots\right)$$
$$\frac{2}{(1-2x)^2}=2+8x+24x^2+\cdots+2^n nx^{n-1}+\cdots$$
$$f'(x)=\frac{2}{(2x-1)^2}=\sum_{n=1}^{\infty}2^n nx^{n-1},-\frac{1}{2}<x<\frac{1}{2}$$

28.
$$f(x)=\frac{1}{x+2}=\sum_{n=0}^{\infty}(-1)^n(x+1)^n,-2<x<0$$
$$\frac{d}{dx}\left(\frac{1}{x+2}\right)=\frac{d}{dx}\left(1-(x+1)+(x+1)^2-(x+1)^3+\cdots+(-1)^n(x+1)^n+\cdots\right)$$
$$-\frac{1}{(x+2)^2}=-1+2(x+1)-3(x+1)^2+\cdots+(-1)^n n(x+1)^{n-1}+\cdots$$
$$f'(x)=-\frac{1}{(x+2)^2}=\sum_{n=1}^{\infty}(-1)^n n(x+1)^{n-1},-2<x<0$$

29.
$$f(x)=\frac{2}{x-1}=\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n(x-3)^n,1<x<5$$
$$\frac{d}{dx}\left(\frac{2}{x-1}\right)=\frac{d}{dx}\left(1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2-\frac{1}{8}(x-3)^3+\cdots+\left(-\frac{1}{2}\right)^n(x-3)^n+\cdots\right)$$
$$-\frac{2}{(x-1)^2}=-\frac{1}{2}+\frac{1}{2}(x-3)-\frac{3}{8}(x-3)^2+\cdots+\left(-\frac{1}{2}\right)^n n(x-3)^{n-1}+\cdots$$
$$f'(x)=-\frac{2}{(x-1)^2}=\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n n(x-3)^{n-1},1<x<5$$

30.
$$f(x)=\frac{6}{3-x}=\sum_{n=0}^{\infty}3\left(\frac{x-1}{2}\right)^n,-1<x<3$$
$$\frac{d}{dx}\left(\frac{6}{3-x}\right)=\frac{d}{dx}\left(3+3\left(\frac{x-1}{2}\right)+3\left(\frac{x-1}{2}\right)^2+\cdots+3\left(\frac{x-1}{2}\right)^n+\cdots\right)$$
$$\frac{6}{(3-x)^2}=\frac{3}{2}+3\left(\frac{x-1}{2}\right)+\cdots+\frac{3}{2}n\left(\frac{x-1}{2}\right)^{n-1}+\cdots$$
$$f'(x)=\frac{6}{(x-3)^2}=\sum_{n=1}^{\infty}\frac{3}{2}n\left(\frac{x-1}{2}\right)^{n-1},-1<x<3$$

31. $f(x)=\dfrac{1}{1-2x}=\sum_{n=0}^{\infty}2^n x^n,\ -\dfrac{1}{2}<x<\dfrac{1}{2}$

The series is centered at 0, that is, $a=0$. We need to find the power series for $\int_0^x f(t)\,dt$.

$$\int_0^x \frac{1}{1-2t}\,dt=\int_0^x (1+2t+4t^2+8t^3+\cdots+2^n t^n+\cdots)\,dt$$

$$-\frac{1}{2}\ln|1-2t|\Big]_0^x=t+t^2+\frac{4}{3}t^3+2t^4+\cdots+\frac{2^n}{n+1}t^{n+1}\Bigg]_0^x$$

$$-\frac{1}{2}\ln|1-2x|=x+x^2+\frac{4}{3}x^3+2x^4+\cdots+\frac{2^n}{n+1}x^{n+1}$$

$$-\frac{1}{2}\ln|2x-1|=\sum_{n=0}^{\infty}\frac{2^n}{n+1}x^{n+1},\ -\frac{1}{2}<x<\frac{1}{2}$$

32. $f(x)=\dfrac{1}{x+2}=\sum_{n=0}^{\infty}(-1)^n(x+1)^n,\ -2<x<0$

The power series is centered at -1, that is, $a=-1$. We need to find the power series for $\int_{-1}^x f(t)\,dt$:

$$\int_{-1}^x \frac{1}{t+2}\,dt=\int_{-1}^x (1-(t+1)+(t+1)^2-(t+1)^3+\cdots+(-1)^n(t+1)^n+\cdots)\,dt$$

$$\ln|t+2|\Big]_{-1}^x=\left[t-\frac{1}{2}(t+1)^2+\frac{1}{3}(t+1)^3-\frac{1}{4}(t+1)^4+\cdots+\frac{(-1)^n}{n+1}(t+1)^{n+1}+\cdots\right]_{-1}^x$$

$$\ln|x+2|=x-(-1)-\frac{1}{2}(x+1)^2+\frac{1}{3}(x+1)^3-\frac{1}{4}(x+1)^4+\cdots+\frac{(-1)^n}{n+1}(x+1)^{n+1}+\cdots$$

$$\ln|x+2|=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(x+1)^{n+1},\ -2<x<0$$

33. $f(x)=\dfrac{2}{x-1}=\sum_{n=0}^{\infty}\left(-\dfrac{1}{2}\right)^n(x-3)^n,\ 1<x<5$

The power series is centered at 3, that is, $a=3$. We need to find the power series for $\int_3^x f(t)\,dt$:

$$\int_3^x \frac{2}{t-1}\,dt=\int_3^x\left(1-\frac{1}{2}(t-3)+\frac{1}{4}(t-3)^2-\frac{1}{8}(t-3)^3+\cdots+\left(-\frac{1}{2}\right)^n(t-3)^n\cdots\right)dt$$

$$2\ln|t-1|\Big]_3^x=\left[t-\frac{1}{4}(t-3)^2+\frac{1}{12}(t-3)^3-\frac{1}{32}(t-3)^4+\cdots+\left(-\frac{1}{2}\right)^n\frac{(t-3)^{n+1}}{n+1}+\cdots\right]_3^x$$

$$2\ln|x-1|-2\ln 2=x-3-\frac{1}{4}(x-3)^2+\frac{1}{12}(x-3)^3-\frac{1}{32}(x-3)^4+\cdots+\left(-\frac{1}{2}\right)^n\frac{(x-3)^{n+1}}{n+1}+\cdots$$

$$2\ln\left|\frac{x-1}{2}\right|=\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\frac{(x-3)^{n+1}}{n+1},\ 1<x<5$$

34. $f(x)=\dfrac{6}{3-x}=\sum_{n=0}^{\infty}3\left(\dfrac{x-1}{2}\right)^n,\ -1<x<3$

The power series is centered at 1, that is, $a=1$. We need to find the power series for $\int_1^x f(t)dt$:

$$\int_1^x \frac{6}{3-t}dt=\int_1^x\left(3+3\left(\frac{t-1}{2}\right)+3\left(\frac{t-1}{2}\right)^2+\cdots+3\left(\frac{t-1}{2}\right)^n+\cdots\right)dt$$

$$-6\ln|3-t|\Big]_1^x=\left[3t+3\left(\frac{t-1}{2}\right)^2+2\left(\frac{t-1}{2}\right)^3+\cdots+\frac{6}{n+1}\left(\frac{t-1}{2}\right)^{n+1}+\cdots\right]_1^x$$

$$-6\ln|3-x|+6\ln 2=3x-3+3\left(\frac{x-1}{2}\right)^2+2\left(\frac{x-1}{2}\right)^3+\cdots+\frac{6}{n+1}\left(\frac{x-1}{2}\right)^{n+1}+\cdots$$

$$-6\ln\left|\frac{3-x}{2}\right|=\sum_{n=0}^{\infty}\frac{6}{n+1}\left(\frac{x-1}{2}\right)^{n+1},\ -1<x<3$$

or

$$6\ln\left(\frac{2}{|x-3|}\right)=\sum_{n=0}^{\infty}\frac{6}{n+1}\left(\frac{x-1}{2}\right)^{n+1},\ -1<x<3$$

35. **(a)** Since the terms are all positive and do not approach zero, the partial sums tend toward infinity.

(b) The partial sums are alternately 1 and 0.

(c) The partial sums alternate between positive and negative while their magnitude increases toward infinity.

36. Since $\sum_{n=0}^{\infty}\dfrac{e^{n\pi}}{\pi^{ne}}=\sum_{n=0}^{\infty}\left(\dfrac{e^{\pi}}{\pi^{e}}\right)^n$, this is a geometric series with common ratio $r=\dfrac{e^{\pi}}{\pi^{e}}\approx 1.03$, which is greater than one.

37. $\sum_{n=0}^{\infty}x^n=20$

$$\frac{1}{1-x}=20,\ |x|<1$$
$$1=20-20x$$
$$20x=19$$
$$x=\frac{19}{20}$$

38. One possible answer:

For any real number $a\neq 0$, use $\dfrac{a}{2}+\dfrac{a}{4}+\dfrac{a}{8}+\dfrac{a}{16}+\dfrac{a}{32}+\cdots$. To get 0, use $1-\dfrac{1}{2}-\dfrac{1}{4}-\dfrac{1}{8}-\dfrac{1}{16}-\dfrac{1}{32}-\cdots$.

39. Assuming the series begins at $n = 1$;

(a) $\sum_{n=1}^{\infty} 2r^{n-1} = \frac{2}{1-r} = 5, \ |r| < 1$

$$2 = 5 - 5r$$
$$5r = 3$$
$$r = \frac{3}{5}$$

Series: $\sum_{n=1}^{\infty} 2\left(\frac{3}{5}\right)^{n-1}$

(b) $\sum_{n=1}^{\infty} \frac{13}{2} r^{n-1} = \frac{\frac{13}{2}}{1-r} = 5, \ |r| < 1$

$$\frac{13}{2} = 5 - 5r$$
$$5r = -\frac{3}{2}$$
$$r = -\frac{3}{10}$$

Series: $\sum_{n=1}^{\infty} \frac{13}{2}\left(-\frac{3}{10}\right)^{n-1}$

40. Let $a = \frac{21}{100}$ and $r = \frac{1}{100}$, giving

$$\begin{aligned} 0.\overline{21} &= 0.21 + 0.21(0.01) + 0.21(0.01)^2 + 0.21(0.01)^3 + \cdots \\ &= \sum_{n=0}^{\infty} 0.21(0.01)^n \\ &= \frac{0.21}{1-0.01} \\ &= \frac{0.21}{0.99} \\ &= \frac{7}{33} \end{aligned}$$

41. Let $a = \frac{234}{1000}$ and $r = \frac{1}{1000}$, giving

$$\begin{aligned} 0.\overline{234} &= 0.234 + 0.234(0.001) + 0.234(0.001)^2 + 0.234(0.001)^3 + \cdots \\ &= \sum_{n=0}^{\infty} 0.234(0.001)^n \\ &= \frac{0.234}{1-0.001} \\ &= \frac{0.234}{0.999} \\ &= \frac{26}{111} \end{aligned}$$

42. $0.\overline{7} = 0.7 + 0.7(0.1) + 0.7(0.1)^2 + 0.7(0.1)^3 + \cdots$

$$= \sum_{n=0}^{\infty} 0.7(0.1)^n = \frac{0.7}{1-0.1} = \frac{0.7}{0.9} = \frac{7}{9}$$

43. $0.\overline{d} = \frac{d}{10} + \frac{d}{10}(0.1) + \frac{d}{10}(0.1)^2 + \frac{d}{10}(0.1)^3 + \cdots$

$$= \sum_{n=0}^{\infty} \frac{d}{10}(0.1)^n = \frac{\frac{d}{10}}{1-0.1} = \frac{d}{10-1} = \frac{d}{9}$$

44. $0.0\overline{6} = 0.06 + 0.06(0.1) + 0.06(0.1)^2 + 0.06(0.1)^3 + \cdots$

$$= \sum_{n=0}^{\infty} 0.06(0.1)^n = \frac{0.06}{1-0.1} = \frac{0.06}{0.9} = \frac{1}{15}$$

45. $1.\overline{414} = 1 + 0.414 + 0.414(0.001) + 0.414(0.001)^2 + \cdots$

$$= 1 + \sum_{n=0}^{\infty} 0.414(0.001)^n = 1 + \frac{0.414}{1-0.001} = 1 + \frac{46}{111} = \frac{157}{111}$$

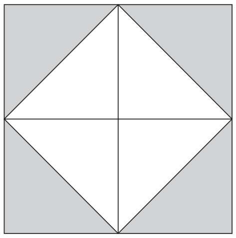
46. $$\begin{aligned}1.24\overline{123} &= 1.24+0.00123+0.00123(0.001)+0.00123(0.001)^2+\cdots\\ &= 1.24+\sum_{n=1}^{\infty}0.00123(0.001)^n\\ &= 1.24+\frac{0.00123}{0.999}\\ &= \frac{124}{100}+\frac{41}{33,300}\\ &= \frac{41,333}{33,300}\end{aligned}$$

47. $$\begin{aligned}3.\overline{142857} &= 3+0.142857(1+0.000001+0.000001^2+\cdots)\\ &= 3+0.142857\sum_{n=0}^{\infty}0.000001^n\\ &= 3+(0.142857)\left(\frac{1}{1-0.000001}\right)\\ &= 3+\frac{0.142857}{0.999999}\\ &= 3+\frac{1}{7}\\ &= \frac{22}{7}\end{aligned}$$

48. $$\begin{aligned}\text{Total distance} &= 4+2[4(0.6)+4(0.6)^2+4(0.6)^3+\cdots]\\ &= 4+2\sum_{n=0}^{\infty}2.4(0.6)^n\\ &= 4+2\cdot\frac{2.4}{1-0.6}\\ &= 4+2\cdot 6\\ &= 16\text{ m}\end{aligned}$$

49. $$\begin{aligned}\text{Total time} &= \sqrt{\frac{4}{4.9}}+2\left[\sqrt{\frac{4(0.6)}{4.9}}+\sqrt{\frac{4(0.6)^2}{4.9}}+\sqrt{\frac{4(0.6)^3}{4.9}}+\ldots\right]\\ &= \sqrt{\frac{4}{4.9}}+2\sqrt{\frac{4(0.6)}{4.9}}\left[1+\sqrt{0.6}+\left(\sqrt{0.6}\right)^2+\ldots\right]\\ &= \sqrt{\frac{4}{4.9}}+2\sqrt{\frac{4(0.6)}{4.9}}\cdot\frac{1}{1-\sqrt{0.6}}\\ &\approx 7.113\text{ sec}\end{aligned}$$

50.



The area of each square is half of the area of the preceding square (see diagram above), so the total of all the areas is $\sum_{n=0}^{\infty} 4\left(\frac{1}{2}\right)^n = \frac{4}{1-\left(\frac{1}{2}\right)} = 8\,\text{m}^2$.

51. Total area $= \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2} \cdot \pi \left(\frac{1}{2^n}\right)^2$

$$= \sum_{n=1}^{\infty} \frac{\pi}{2} \cdot \left(\frac{1}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{\pi}{4}\left(\frac{1}{2}\right)^n$$

$$= \frac{\frac{\pi}{4}}{1-\left(\frac{1}{2}\right)}$$

$$= \frac{\pi}{2}$$

52. **(a)** $S - rS = (a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1}) - (ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n)$

$$= a - ar^n$$

(b) Factor and divide by $1 - r$:

$$S - rS = a - ar^n$$

$$S(1-r) = a - ar^n$$

$$S = \frac{a - ar^n}{1-r}$$

53. Using the notation $S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$, the formula from Exercise 52 is $S_n = \frac{a - ar^n}{1-r}$.

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$ and so $\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^n}{1-r} = \frac{a}{1-r}$.

If $|r| > 1$ or $r = -1$, then r^n has no finite limit as $n \to \infty$, so the expression $\frac{a - ar^n}{1-r}$ has no finite limit and $\sum_{n=1}^{\infty} ar^{n-1}$ diverges. If $r = 1$, then the nth partial sum is na, which goes to $\pm\infty$, depending on the sign of a.

54. Comparing $\frac{1}{1+3x}$ with $\frac{a}{1-r}$, the leading term is $a = 1$ and the common ratio is $r = -3x$.

Series: $1 - 3x + 9x^2 - \cdots + (-3x)^n + \cdots$

Interval: The series converges when $|-3x| < 1$, so the interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

55. Comparing $\frac{x}{1-2x}$ with $\frac{a}{1-r}$, the first term is $a = x$ and the common ratio is $r = 2x$.

Series: $x + 2x^2 + 4x^3 + \cdots + 2^{n-1}x^n + \cdots$

Interval: The series converges when $|2x| < 1$, so the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

56. Comparing $r = x^3$. with $\frac{a}{1-r}$, the first term is $a = 3$ and the common ratio is $r = x^3$.

Series: $3 + 3x^3 + 3x^6 + \cdots + 3x^{3n} + \cdots$

Interval: The series converges when $\left|x^3\right| < 1$, so the interval of convergence is (–1, 1).

57. Comparing $\frac{1}{1+(x-4)}$ with $\frac{a}{1-r}$, the first term is $a = 1$ and the common ratio is $r = -(x-4)$.

Series: $1 - (x-4) + (x-4)^2 - \cdots + (-1)^n(x-4)^n + \cdots$

Interval: The series converges when $|x-4| < 1$, so the interval of convergence is (3, 5).

58. Comparing $\frac{1}{4}\left(\frac{1}{1+(x-1)}\right)$ with $\frac{a}{1-r}$, the first term is $a = \frac{1}{4}$ and the common ratio is $r = -(x-1)$.

Series: $\frac{1}{4} - \frac{1}{4}(x-1) + \frac{1}{4}(x-1)^2 - \cdots + \frac{1}{4}(-1)^n(x-1)^n + \cdots$

Interval: The series converges when $|x-1| < 1$, so the interval of convergence is (0, 2).

59. Rewriting $\frac{1}{2-x}$ as $\frac{1}{1-(x-1)}$ and comparing with $\frac{a}{1-r}$, the first term is $a = 1$ and the common ratio is $r = x - 1$.

Series: $1 + (x-1) + (x-1)^2 + \cdots + (x-1)^n + \cdots$

Interval: The series converges when $|x-1| < 1$, so the interval of convergence is (0, 2).

Alternate solution: Rewriting $\frac{1}{2-x}$ as $\frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right)$ and comparing with $\frac{a}{1-r}$, the first term is $a = \frac{1}{2}$ and the common ratio is $r = \frac{x}{2}$.

Series: $\frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \cdots + \frac{1}{2^{n+1}}x^n + \cdots$

Interval: The series converges when $\left|\frac{x}{2}\right| < 1$, so the interval of convergence is (–2, 2).

60. $$\begin{aligned} 1+e^b+e^{2b}+e^{3b}+\cdots &= \sum_{n=0}^{\infty}(e^b)^n \\ &= \frac{1}{1-e^b}=9 \\ 1 &= 9-9e^b \\ 9e^b &= 8 \\ e^b &= \frac{8}{9} \\ b &= \ln\left(\frac{8}{9}\right) \end{aligned}$$

61. **(a)** When $t=1$, $S=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n=\frac{1}{1-\left(\frac{1}{2}\right)}=2.$

(b) S converges when $\left|\frac{t}{t+1}\right|<1$

Equivalently: $\left|\frac{t+1}{t}\right|>1$

$$\left|1+\frac{1}{t}\right|>1$$

$$\begin{array}{lcl} 1+\frac{1}{t}<-1 & \text{or} & 1+\frac{1}{t}>1 \\ \frac{1}{t}<-2 & & \frac{1}{t}>0 \\ t>-\frac{1}{2} & & t>0 \end{array}$$

Thus, S converges for all $t>-\frac{1}{2}.$

(c) For $t>-\frac{1}{2}$, we have $$\begin{aligned} S &= \sum_{n=0}^{\infty}\left(\frac{t}{1+t}\right)^n \\ &= \frac{1}{1-\frac{t}{1+t}} \\ &= \frac{1+t}{(1+t)-t} \\ &= 1+t, \end{aligned}$$

so $S>10$ when $t>9$.

62. **(a)** The series converges to S means that $\lim_{n\to\infty} S_n=S$, where $S_n=\sum_{k=1}^{k=n}a_k$ is the nth partial sum of the series.

(b) $S_n=(a_1+\cdots+a_{n-1})+a_n=S_{n-1}+a_n$

(c) $\lim_{n\to\infty} S_n=\lim_{n\to\infty} S_{n-1}+\lim_{n\to\infty} a_n$ so $S=S+\lim_{n\to\infty} a_n$ or $\lim_{n\to\infty} a_n=0.$

63. Since $\frac{1}{x} = 1-(x-1)+(x-1)^2-(x-1)^3+\cdots+(-1)^n(x-1)^n+\cdots,$

we may write: $\ln x = \int_1^x \frac{1}{t}\,dt = x-1-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots+\frac{(-1)^n(x-1)^{n+1}}{n+1}.$

64. To determine our starting point, we note that $\int f(x)dx = \int 2(1-x)^{-3}dx = (1-x)^{-2}+C.$ Using the result from Example 4, we have:

$$(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots+nx^{n-1}+\cdots$$
$$\frac{d}{dx}(1-x)^{-2} = \frac{d}{dx}(1+2x+3x^2+4x^3+\cdots+nx^{n-1}+\cdots)$$
$$2(1-x)^{-3} = 0+2+6x+12x^2+\cdots+n(n-1)x^{n-2}+\cdots$$

Thus, $f(x) = 2+6x+12x^2+\cdots+n(n-1)x^{n-2}+\cdots.$

Replacing n by $n + 2$, this may be written as $f(x) = 2+6x+12x^2+(n+2)(n+1)x^n+\cdots.$

Interval: The series converges when $|x|<1,$ so the interval of convergence is (–1, 1).

65. **(a)** No, because if you differentiate it again, you would have the original series for *f*, but by Theorem 1, that would have to converge for $-2 < x < 2$, which contradicts the assumption that the original series converges only for $-1 < x < 1$.

(b) No, because if you integrate it again, you would have the original series for *f*, but by Theorem 2, that would have to converge for $-2 < x < 2$, which contradicts the assumption that the original series converges only for $-1 < x < 1$.

66. False; it diverges because it is a geometric series with ratio $r = 1.01 > 1$.

67. False; it converges because it is a geometric series with ratio $r = \frac{1}{2} < 1.$

68. C; $\frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$

69. A. Converges for $|x-1|<1,$ or $0<x<2.$

70. E; $\frac{1}{1-(x-1)} = \frac{1}{2-x}$

71. D; the function must be the antiderivative of $\frac{1}{2-x}$ that passes through the point (1, 0), and that function is $-\ln(2-x)$.

72. **(a)** Comparing $f(t) = \frac{4}{1+t^2}$ with $\frac{a}{1-r},$ the first term is $a = 4$ and the common ratio is $r = -t^2$.

First four terms: $4-4t^2+4t^4-4t^6$

General term: $(-1)^n(4t^{2n})$

(b) Note that $G(0) = 0$, so the constant term of the power series for $G(x)$ will be 0. Integrate the terms for $f(x)$ to obtain the terms for $G(x)$.

First four terms: $4x - \frac{4}{3}x^3 + \frac{4}{5}x^5 - \frac{4}{7}x^7$

General term: $(-1)^n\left(\frac{4}{2n+1}\right)x^{2n+1}$

(c) The series in part (a) converges when $\left|-t^2\right| < 1$, so the interval of convergence is $(-1, 1)$.

(d) The two numbers are $x = \pm 1$, which result in the convergent series

$G(1) = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots + (-1)^n\left(\frac{4}{2n+1}\right) + \cdots$ and $G(-1) = -4 + \frac{4}{3} - \frac{4}{5} + \frac{4}{7} - \cdots + (-1)^{n-1}\left(\frac{4}{2n+1}\right) + \cdots$,

respectively.

73. Let $L = \lim_{n\to\infty} a_n$. Then by definition of convergence, for $\frac{\varepsilon}{2}$ there corresponds an N such that for all m and n,

$n, m > N \Rightarrow |a_m - L| < \frac{\varepsilon}{2}$ and $|a_n - L| < \frac{\varepsilon}{2}$. Now,

$$\begin{aligned} |a_m - a_n| &= |a_m - L + L - a_n| \\ &\le |a_m - L| + |a_n - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

whenever $m > N$ and $n > N$.

74. Given an $\varepsilon > 0$, by definition of convergence there corresponds an N such that for all $n > N$, $|L_1 - a_n| < \varepsilon$ and $|L_2 - a_n| < \varepsilon$. (There is one such number for each series, and we may let N be the larger of the two numbers.) Now

$$\begin{aligned} |L_2 - L_1| &= |L_2 - a_n + a_n - L_1| \\ &\le |L_2 - a_n| + |a_n - L_n| \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon. \end{aligned}$$

$|L_2 - L_1| < 2\epsilon$ says that the difference between two fixed values is smaller than any positive number 2ϵ. The only nonnegative number smaller than every positive number is 0, so $|L_2 - L_1| = 0$ or $L_1 = L_2$.

75. Consider the two subsequences $a_{k(n)}$ and $a_{i(n)}$, where $\lim_{n\to\infty} a_{k(n)} = L_1$, $\lim_{n\to\infty} a_{i(n)} = L_2$, and $L_1 \ne L_2$. Given an $\varepsilon > 0$ there corresponds an N_1 such that for $k(n) > N_1$, $|a_{k(n)} - L_1| < \varepsilon$, and an N_2 such that for $i(n) > N_2$, $|a_{i(n)} - L_2| < \varepsilon$. Assume a_n converges.

Let $N = \max\{N_1, N_2\}$. Then for $n > N$, we have that $|a_n - L_1| < \epsilon$ and $|a_n - L_2| < \varepsilon$ for infinitely many n. This implies that $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$ where $L_1 \ne L_2$. Since the limit of a sequence is unique (by Exercise 74), a_n does not converge and hence diverges.

76. **(a)** $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(3n+1)}{(n+1)} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$

$$= \lim_{n\to\infty} \frac{3+\frac{1}{n}}{1+\frac{1}{n}}$$

$$= \frac{3+0}{1+0}$$

$$= 3$$

(b) We showed in part (a) that $\lim_{n\to\infty} a_n = 3.$

A similar calculation would show that

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x+1}{x+1} = 3$

This tells us that as x gets infinitely large, the graph $y = f(x)$ gets infinitely close to the line $y = 3$. This is exactly what it means for $y = 3$ to be a horizontal asymptote of the graph.

Section 10.2 Taylor Series (pp. 488–498)

Exploration 1 Designing a Polynomial to Specifications

1. Since $P(0) = 5$, we know that the constant coefficient is 5. Since $P'(0) = 7$ we know that the coefficient of x is 7. Since $P''(0) = 11$ we know that the coefficient of x^2 is $\frac{11}{2}$. (The 2 in the denominator is needed to cancel the factor of 2 that results from differentiating x^2.) Similarly, we find the coefficients of x^3 and x^4 to be $\frac{13}{6}$ and $\frac{17}{24}$.

Thus, $P(x) = 5 + 7x + \frac{11}{2}x^2 + \frac{13}{6}x^3 + \frac{17}{24}x^4.$

Exploration 2 A Power Series for the Cosine

1. $\cos(0) = 1$

$\cos'(0) = -\sin(0) = 0$

$\cos''(0) = -\cos(0) = -1$

$\cos^{(3)}(0) = \sin(0) = 0$

etc.

The pattern 1, 0, –1, 0 will repeat forever.

Therefore, $P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$, and the Taylor series is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots.$$

2. A clever shortcut is simply to differentiate the previously-discovered series for sin x term-by-term!

Exploration 3 Approximating sin 13

1. 0.4201670368…

4. 20 terms.

Quick Review 10.2

1. $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

$f'''(x) = 8e^{2x}$

$f^{(n)}(x) = 2^n e^{2x}$

2. $f(x) = \frac{1}{x-1}$

$f'(x) = -(x-1)^{-2}$

$f''(x) = 2(x-1)^{-3}$

$f'''(x) = -6(x-1)^{-4}$

$f^{(n)}(x) = (-1)^n n!(x-1)^{-(n+1)}$

3. $f(x) = 3^x$

$f'(x) = 3^x \ln 3$

$f''(x) = 3^x (\ln 3)^2$

$f'''(x) = 3^x (\ln 3)^3$

$f^{(n)}(x) = 3^x (\ln 3)^n$

4. $f(x) = \ln(x)$

$f'(x) = x^{-1}$

$f''(x) = -x^{-2}$

$f'''(x) = 2x^{-3}$

$f^{(4)}(x) = -6x^{-4}$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,x^{-n}$ for $n \ge 1$

5. $f(x) = x^n$

$f'(x) = nx^{n-1}$

$f''(x) = n(n-1)x^{n-2}$

$f'''(x) = n(n-1)(n-2)x^{n-3}$

$f^{(k)}(x) = \dfrac{n!}{(n-k)!}x^{n-k}$

$f^{(n)}(x) = \dfrac{n!}{0!}x^0 = n!$

6. $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{x^n}{n!} = \dfrac{nx^{n-1}}{n!} = \dfrac{x^{n-1}}{(n-1)!}$

7. $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{2^n(x-a)^n}{n!}$

$= \dfrac{2^n n(x-a)^{n-1}}{n!}$

$= \dfrac{2^n(x-a)^{n-1}}{(n-1)!}$

8. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left[(-1)^n\dfrac{x^{2n+1}}{(2n+1)!}\right]$

$= (-1)^n\dfrac{(2n+1)x^{2n}}{(2n+1)!}$

$= \dfrac{(-1)^n x^{2n}}{(2n)!}$

9. $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{(x+a)^{2n}}{(2n)!}$

$= \dfrac{2n(x+a)^{2n-1}}{(2n)!}$

$= \dfrac{(x+a)^{2n-1}}{(2n-1)!}$

10. $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{(1-x)^n}{n!}$

$= \dfrac{n(1-x)^{n-1}(-1)}{n!}$

$= -\dfrac{(1-x)^{n-1}}{(n-1)!}$

Section 10.2 Exercises

1. $f(0) = \sqrt{1+x^2}\Big|_{x=0} = 1$

$f'(0) = \dfrac{x}{\sqrt{1+x^2}}\Big|_{x=0} = 0$

$f''(0) = \dfrac{1}{(1+x^2)^{3/2}}\Big|_{x=0} = 1$

$f'''(0) = \dfrac{-3x}{(1+x^2)^{5/2}}\Big|_{x=0} = 0$

$f^{(4)}(0) = \dfrac{12x^2-3}{(1+x^2)^{7/2}}\Big|_{x=0} = -3$

$P_4(x) = 1 + 0x + \dfrac{1}{2}x^2 + \dfrac{0}{6}x^3 + \dfrac{-3}{24}x^4$

$= 1 + \dfrac{1}{2}x^2 - \dfrac{1}{8}x^4$

2. $f(0) = e^{2x}\Big|_{x=0} = 1$

$f'(0) = 2e^{2x}\Big|_{x=0} = 2$

$f''(0) = 4e^{2x}\Big|_{x=0} = 4$

$f'''(0) = 8e^{2x}\Big|_{x=0} = 8$

$f^{(4)}(0) = 16e^{2x}\Big|_{x=0} = 16$

$P_4(x) = 1 + 2x + 2x^2 + \dfrac{4}{3}x^3 + \dfrac{2}{3}x^4$

3. $f(0) = \dfrac{1}{x+2}\Big|_{x=0} = \dfrac{1}{2}$

$f'(0) = \dfrac{-1}{(x+2)^2}\Big|_{x=0} = -\dfrac{1}{4}$

$f''(0) = \dfrac{2}{(x+2)^3}\Big|_{x=0} = \dfrac{1}{4}$

$f'''(0) = -\dfrac{-6}{(x+2)^4}\Big|_{x=0} = -\dfrac{3}{8}$

$f^{(4)}(0) = \dfrac{24}{(x+2)^5}\Big|_{x=0} = \dfrac{3}{4}$

$f^{(5)}(0) = \dfrac{-120}{(x+2)^6}\Big|_{x=0} = -\dfrac{15}{8}$

$$P_5(x) = \frac{1}{2} - \frac{1}{4}x + \frac{1}{4}\cdot\frac{1}{2}x^2 - \frac{3}{8}\cdot\frac{1}{6}x^3 + \frac{3}{4}\cdot\frac{1}{24}x^4 - \frac{15}{8}\cdot\frac{1}{120}x^5$$
$$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 + \frac{1}{32}x^4 - \frac{1}{64}x^5$$
$$\frac{1}{x+2} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)^{n+1} x^n$$

4. $$f(0) = e^{1-x}\Big|_{x=0} = e$$
$$f'(0) = -e^{1-x}\Big|_{x=0} = -e$$
$$f''(0) = e^{1-x}\Big|_{x=0} = e$$
$$f'''(0) = -e^{1-x}\Big|_{x=0} = -e$$
$$f^{(4)}(0) = e^{1-x}\Big|_{x=0} = e$$
$$f^{(5)}(0) = -e^{1-x}\Big|_{x=0} = -e$$
$$P_5(x) = e - ex + \frac{e}{2}x^2 - \frac{e}{6}x^3 + \frac{e}{24}x^4 - \frac{e}{120}x^5$$
$$e^{1-x} = \sum_{n=0}^{\infty} (-1)^n \frac{e}{n!} x^n$$

5. Substitute $2x$ for x in the Maclaurin series for $\sin x$ shown at the end of Section 10.2.
$$\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots + (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} + \cdots$$
$$= 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \cdots + \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} + \cdots$$
This series converges for all real x.

6. Substitute $-x$ for x in the Maclaurin series for $\ln(1 + x)$ shown at the end of Section 10.2.
$$\ln(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \cdots + (-1)^{n-1}\frac{(-x)^n}{n} + \cdots$$
$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \cdots$$
This series converges when $-1 < -x \le 1$ or $-1 \le x < 1$, so the interval of convergence is $[-1, 1)$.

7. Substitute x^2 for x in the Maclaurin series for $\tan^{-1} x$ shown at the end of Section 10.2.
$$\tan^{-1} x^2 = x^2 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \cdots + (-1)^n \frac{(x^2)^{2n+1}}{2n+1} + \cdots$$
$$= x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \cdots + \frac{(-1)^n x^{4n+2}}{2n+1} + \cdots$$
This series converges when $\left|x^2\right| \le 1$, so the interval of convergence is $[-1, 1]$.

8. $7xe^x = 7x\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right)$

$$= 7x + 7x^2 + \frac{7x^3}{2!} + \cdots + \frac{7x^{n+1}}{n!}$$

This series converges for all real x.

9. Substitute $\frac{x}{2}$ for x in the Maclaurin series for $\cos x$ shown at the end of Section 10.2.

$$\cos\left(\frac{x}{2}\right) = 1 - \frac{\left(\frac{x}{2}\right)^2}{2!} + \frac{\left(\frac{x}{2}\right)^4}{4!} - \cdots + (-1)^n \frac{\left(\frac{x}{2}\right)^{2n}}{(2n)!} + \cdots$$

$$= 1 - \frac{x^2}{2^2 \cdot 2!} + \frac{x^4}{2^4 \cdot 4!} - \cdots + (-1)^n \frac{x^{2n}}{2^{2n} \cdot (2n)!} + \cdots$$

This series converges for all real x.

10. $x^2 \cos x = x^2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots\right)$

$$= x^2 - \frac{x^4}{2} + \frac{x^6}{24} - \cdots + \frac{(-1)^n x^{2n+2}}{(2n)!} + \cdots$$

The series converges for all real x.

11. Factor out x and substitute x^3 for x in the Maclaurin series for $\frac{1}{1-x}$ shown at the end of Section 10.2.

$$\frac{x}{1-x^3} = x\left(\frac{1}{1-x^3}\right)$$

$$= x[1 + x^3 + (x^3)^2 + \cdots + (x^3)^n + \cdots]$$

$$= x + x^4 + x^7 + \cdots + x^{3n+1} + \cdots$$

The series converges for $\left|x^3\right| < 1$, so the interval of convergence is $(-1, 1)$.

12. Substitute $-2x$ for x in the Maclaurin series for e^x shown at the end of Section 10.2.

$$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \cdots + \frac{(-2x)^n}{n!} + \cdots$$

$$= 1 - 2x + 2x^2 - \cdots + \frac{(-1)^n 2^n x^n}{n!} + \cdots$$

The series converges for all real x.

13. $f(2)=\left.\frac{1}{x+1}\right|_{x=2}=\frac{1}{3}$

$f'(2)=\left.\frac{-1}{(x+1)^2}\right|_{x=2}=-\frac{1}{9}$

$f''(2)=\left.\frac{2}{(x+1)^3}\right|_{x=2}=\frac{2}{27}$

$\vdots$

$f^{(n)}(2)=\left.\frac{(-1)^n n!}{(x+1)^{n+1}}\right|_{x=2}=\frac{(-1)^n n!}{3^{n+1}}$

$$\begin{aligned}\frac{1}{x+1}&=\frac{1}{3}-\frac{1}{9}(x-2)+\frac{2}{27}\frac{(x-2)^2}{2!}+\cdots+\frac{(-1)^n n!}{3^{n+1}}\frac{(x-2)^n}{n!}+\cdots\\&=\frac{1}{3}-\frac{1}{9}(x-2)+\frac{1}{27}(x-2)^2+\cdots+\frac{(-1)^n}{3^{n+1}}(x-2)^n+\cdots\\&=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-2)^n\end{aligned}$$

14. $f(1)=\left.e^{x/2}\right|_{x=1}=e^{1/2}$

$f'(1)=\left.\frac{1}{2}e^{x/2}\right|_{x=1}=\frac{1}{2}e^{1/2}$

$f''(1)=\left.\frac{1}{4}e^{x/2}\right|_{x=1}=\frac{1}{4}e^{1/2}$

$\vdots$

$f^{(n)}(1)=\left(\frac{1}{2}\right)^n \left.e^{x/2}\right|_{x=1}=\left(\frac{1}{2}\right)^n e^{1/2}$

$$\begin{aligned}e^{x/2}&=e^{1/2}+\frac{1}{2}e^{1/2}(x-1)+\frac{1}{4}e\cdot\frac{(x-1)^2}{2}+\cdots+\left(\frac{1}{2}\right)^n e^{1/2}\cdot\frac{(x-1)^n}{n!}+\cdots\\&=e^{1/2}+e^{1/2}\frac{(x-1)}{2}+e^{1/2}\frac{(x-1)^2}{8}+\cdots+e^{1/2}\frac{(x-1)^n}{2^n n!}+\cdots\\&=\sum_{n=0}^{\infty}e^{1/2}\frac{(x-1)^n}{2^n n!}\end{aligned}$$

15. **(a)** Since f is a cubic polynomial, it is its own Taylor polynomial of order 3.

$P_3(x)=x^3-2x+4=4-2x+x^3$

(b) $f(1)=\left.x^3-2x+4\right|_{x=1}=3$

$f'(1)=\left.3x^2-2\right|_{x=1}=1$

$f''(1)=\left.6x\right|_{x=1}=6$, so $\frac{f''(1)}{2!}=3$

$f'''(1)=\left.6\right|_{x=1}=6$, so $\frac{f'''(1)}{3!}=1$

$P_3(x)=3+(x-1)+3(x-1)^2+(x-1)^3$

16. **(a)** Since f is a cubic polynomial, it is its own Taylor polynomial of order 3.

$$P_3(x) = 2x^3 + x^2 + 3x - 8$$
$$= -8 + 3x + x^2 + 2x^3$$

(b) $f(1) = 2x^3 + x^2 + 3x - 8\Big|_{x=1} = -2$

$f'(1) = 6x^2 + 2x + 3\Big|_{x=1} = 11$

$f''(1) = 12x + 2\Big|_{x=1} = 14$, so $\dfrac{f''(1)}{2!} = 7$

$f'''(1) = 12\Big|_{x=1} = 12$, so $\dfrac{f'''(1)}{3!} = 2$

$P_3(x) = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3$

17. **(a)** Since $f(0) = f'(0) = f''(0) = f'''(0) = 0$, the Taylor polynomial of order 3 is $P_3(0) = 0$.

(b) $f(1) = x^4\Big|_{x=1} = 1$

$f'(1) = 4x^3\Big|_{x=1} = 4$

$f''(1) = 12x^2\Big|_{x=1} = 12$, so $\dfrac{f''(1)}{2!} = 6$

$f'''(1) = 24x\Big|_{x=1} = 24$, so $\dfrac{f'''(1)}{3!} = 4$

$P_3(x) = 1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3$

18. $f(2) = \dfrac{1}{x}\Big|_{x=2} = \dfrac{1}{2}$

$f'(2) = -x^{-2}\Big|_{x=2} = -\dfrac{1}{4}$

$f''(2) = 2x^{-3}\Big|_{x=2} = \dfrac{1}{4}$, so $\dfrac{f''(2)}{2!} = \dfrac{1}{8}$

$f'''(2) = -6x^{-4}\Big|_{x=2} = -\dfrac{3}{8}$, so $\dfrac{f'''(2)}{3!} = -\dfrac{1}{16}$

$P_0(x) = \dfrac{1}{2}$

$P_1(x) = \dfrac{1}{2} - \dfrac{x-2}{4}$

$P_2(x) = \dfrac{1}{2} - \dfrac{x-2}{4} + \dfrac{(x-2)^2}{8}$

$P_3(x) = \dfrac{1}{2} - \dfrac{x-2}{4} + \dfrac{(x-2)^2}{8} - \dfrac{(x-2)^3}{16}$

19. $f\left(\frac{\pi}{4}\right) = \sin x\Big|_{x=\pi/4} = \frac{\sqrt{2}}{2}$

$f'\left(\frac{\pi}{4}\right) = \cos x\Big|_{x=\pi/4} = \frac{\sqrt{2}}{2}$

$f''\left(\frac{\pi}{4}\right) = -\sin x\Big|_{x=\pi/4} = -\frac{\sqrt{2}}{2}$,

so $\frac{f''\left(\frac{\pi}{4}\right)}{2!} = -\frac{\sqrt{2}}{4}$.

$f'''\left(\frac{\pi}{4}\right) = -\cos x\Big|_{x=\pi/4} = -\frac{\sqrt{2}}{2}$,

so $\frac{f'''\left(\frac{\pi}{4}\right)}{3!} = -\frac{\sqrt{2}}{12}$.

$P_0(x) = \frac{\sqrt{2}}{2}$

$P_1(x) = \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{4}\right)$

$P_2(x) = \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{4}\right) - \left(\frac{\sqrt{2}}{4}\right)\left(x - \frac{\pi}{4}\right)^2$

$P_3(x) = \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{4}\right) - \left(\frac{\sqrt{2}}{4}\right)\left(x - \frac{\pi}{4}\right)^2 - \left(\frac{\sqrt{2}}{12}\right)\left(x - \frac{\pi}{4}\right)^3$

20. $f\left(\frac{\pi}{4}\right) = \cos x\Big|_{x=\pi/4} = \frac{\sqrt{2}}{2}$

$f'\left(\frac{\pi}{4}\right) = -\sin x\Big|_{x=\pi/4} = -\frac{\sqrt{2}}{2}$

$f''\left(\frac{\pi}{4}\right) = -\cos x\Big|_{x=\pi/4} = -\frac{\sqrt{2}}{2}$,

so $\frac{f''\left(\frac{\pi}{4}\right)}{2!} = -\frac{\sqrt{2}}{4}$

$f'''\left(\frac{\pi}{4}\right) = \sin x\Big|_{x=\pi/4} = \frac{\sqrt{2}}{2}$,

so $\frac{f'''\left(\frac{\pi}{4}\right)}{3!} = \frac{\sqrt{2}}{12}$

$P_0(x) = \frac{\sqrt{2}}{2}$

$P_1(x) = \frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{4}\right)$

$P_2(x) = \frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{4}\right) - \left(\frac{\sqrt{2}}{4}\right)\left(x - \frac{\pi}{4}\right)^2$

$P_3(x) = \frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{4}\right) - \left(\frac{\sqrt{2}}{4}\right)\left(x - \frac{\pi}{4}\right)^2 + \left(\frac{\sqrt{2}}{12}\right)\left(x - \frac{\pi}{4}\right)^3$

21. $f(4) = x^{1/2}\Big|_{x=4} = 2$

$f'(4) = \frac{1}{2}x^{-1/2}\Big|_{x=4} = \frac{1}{4}$

$f''(4) = -\frac{1}{4}x^{-3/2}\Big|_{x=4} = -\frac{1}{32},$

so $\frac{f''(4)}{2!} = -\frac{1}{64}$

$f'''(4) = \frac{3}{8}x^{-5/2}\Big|_{x=4} = \frac{3}{256},$ so $\frac{f'''(4)}{3!} = \frac{1}{512}$

$P_0(x) = 2$

$P_1(x) = 2 + \frac{x-4}{4}$

$P_2(x) = 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64}$

$P_3(x) = 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512}$

22. (a) $P_3(x) = 4 + 5x + \frac{-8}{2!}x^2 + \frac{6}{3!}x^3$

$= 4 + 5x - 4x^2 + x^3$

$f(0.2) \approx P_3(0.2) = 4.848$

(b) Since the Taylor series of $f'(x)$ can be obtained by differentiating the terms of the Taylor series of $f(x)$, the second order Taylor polynomial of $f'(x)$ is given by $5 - 8x + 3x^2$. Evaluating at $x = 0.2$, $f'(0.2) \approx 3.52$

23. (a) $P_3(x)$

$= 4 + (-1)(x-1) + \frac{3}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$

$= 4 - (x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

$f(1.2) \approx P_3(1.2) \approx 3.863$

(b) Since the Taylor series of $f'(x)$ can be obtained by differentiating the terms of the Taylor series of $f(x)$, the second order Taylor polynomial of $f'(x)$ is given by $-1 + 3(x-1) + (x-1)^2$. Evaluating at $x = 1.2$, $f'(1.2) \approx -0.36$

24. (a) Since $f'(0)x = \frac{x}{2!}$, $f'(0) = \frac{1}{2!} = \frac{1}{2}$.

Since $\frac{f^{(10)}(0)}{10!}x^{10} = \frac{x^{10}}{11!}$,

$f^{(10)}(0) = \frac{10!}{11!} = \frac{1}{11}$.

(b) Multiply each term of $f(x)$ by x.

$g(x)$

$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots$

(c) $g(x) = e^x - 1$

25. (a) Substitute $\frac{x}{2}$ for x in the Maclaurin series for e^x shown at the end of Section 10.2

$e^{x/2} = 1 + \frac{x}{2} + \frac{\left(\frac{x}{2}\right)^2}{2} + \cdots + \frac{\left(\frac{x}{2}\right)^n}{n!} + \cdots$

$= 1 + \frac{x}{2} + \frac{x^2}{8} + \cdots + \frac{x^n}{2^n \cdot n!}$

(b) $g(x) = \frac{e^x - 1}{x}$

$= \frac{1}{x}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right) - 1\right]$

$= \frac{1}{x}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right)$

$= 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots + \frac{x^{n-1}}{n!} + \cdots$

This can also be written as

$1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots + \frac{x^n}{(n+1)!} + \cdots$.

(c) $g'(x) = \frac{d}{dx}\left(\frac{e^x - 1}{x}\right)$

$= \frac{(x)(e^x) - (e^x - 1)(1)}{x^2}$

$= \frac{xe^x - e^x + 1}{x^2}$

$g'(1) = \frac{e - e + 1}{1} = 1$

From the series,

$$g'(x) = \frac{d}{dx}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots+\frac{x^n}{(n+1)!}+\cdots\right)$$
$$= \frac{1}{2!}+\frac{2x}{3!}+\frac{3x^2}{4!}+\cdots+\frac{nx^{n-1}}{(n+1)!}+\cdots$$
$$= \sum_{n=1}^{\infty}\frac{nx^{n-1}}{(n+1)!}$$

Therefore, $g'(1) = \sum_{n=1}^{\infty}\frac{n}{(n+1)!}$, which means $\sum_{n=1}^{\infty}\frac{n}{(n+1)!} = 1$.

26. **(a)** Factor out 2 and substitute t^2 for x in the Maclaurin series for $\frac{1}{1-x}$ shown at the end of Section 10.2.

$$f(t) = \frac{2}{1-t^2}$$
$$= 2\left(\frac{1}{1-t^2}\right)$$
$$= 2\left[1+t^2+(t^2)^2+(t^2)^3+\cdots+(t^2)^n+\cdots\right]$$
$$= 2+2t^2+2t^4+2t^6+\cdots+2t^{2n}+\cdots$$

(b) Since $G(0)=0$, the constant term is zero and we may find $G(x)$ by integrating the terms of the series for $f(x)$.

$$G(x) = 2x+\frac{2x^3}{3}+\frac{2x^5}{5}+\frac{2x^7}{7}+\cdots+\frac{2x^{2n+1}}{2n+1}+\cdots$$

27. **(a)** $f(0) = (1+x)^{1/2}\Big|_{x=0} = 1$

$$f'(0) = \frac{1}{2}(1+x)^{-1/2}\Big|_{x=0} = \frac{1}{2}$$

$$f''(0) = -\frac{1}{4}(1+x)^{-3/2}\Big|_{x=0} = -\frac{1}{4},$$

so $\frac{f''(0)}{2!} = -\frac{1}{8}$

$$f'''(0) = \frac{3}{8}(1+x)^{-5/2}\Big|_{x=0} = \frac{3}{8},$$

so $\frac{f'''(0)}{3!} = \frac{1}{16}$

$$P_4(x) = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}$$

(b) Since $g(x) = f(x^2)$, the first four terms are $1+\frac{x^2}{2}-\frac{x^4}{8}+\frac{x^6}{16}$.

(c) Since $h(0)=5$, the constant term is 5. The next three terms are obtained by integrating the first three terms of the answer to part (b). The first four terms of the series for $h(x)$ are $5+x+\frac{x^3}{6}-\frac{x^5}{40}$.

28. **(a)** $a_0 = 1$

$$a_1 = \frac{3}{1}a_0 = 3\cdot 1 = 3$$
$$a_2 = \frac{3}{2}a_1 = \frac{3}{2}\cdot 3 = \frac{9}{2}$$
$$a_3 = \frac{3}{3}a_2 = a_2 = \frac{9}{2}$$

Since each term is obtained by multiplying the previous term by

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = (n+1)^2\left(\frac{2}{3}\right)^{n+1}\cdot\frac{1}{n^2}\left(\frac{3}{2}\right)^n$$
$$= \frac{2}{3} < 1.$$

$$\sum_{n=0}^{\infty}a_nx^n$$
$$= 1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3+\cdots+\frac{3^n}{n!}x^n+\cdots$$

(b) Since the series can be written as $\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}$, it represents the function $f(x) = e^{3x}$.

(c) $f'(1) = 3e^{3x}\Big|_{x=1} = 3e^3$

29. First, note that cos 18 ≈ 0.6603.

Using $\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$, enter the following two-step commands on your home screen and continue to hit ENTER.

```
0→N:1→T
                      1
N+1→N:T+(-1)^N*1
8^(2N)/(2N)!→T
                   -161
                   4213
               -43026.2
```

The sum corresponding to $N = 25$ is about 0.6582 (not within 0.001 of the exact value),

and the sum corresponding to $N = 26$ is about 0.6606, which is within 0.001 of the exact value. Since we began with $N = 0$, it takes a total of 27 terms (or, up to and including the 52nd degree term).

30. One possible answer: Because the end behavior of a polynomial must be unbounded and $\sin x$ is not unbounded. Another: Because $\sin x$ has an infinite number of local extrema, but a polynomial can have only a finite number.

31. (1) Since $\sin 0 = 0$, the series for $\sin x$ is the series that begins with x. Also, $\sin x$ is an odd function, so the series for $\sin x$ will contain only odd powers of x.

(2) Since $\cos 0 = 1$, the series for $\cos x$ is the series that begins with 1. Also, $\cos x$ is an even function, so the series for $\cos x$ will contain only even powers of x.

32. Replace x by $3x$ in series for $\sin x$. Therefore, we have $\frac{(3x)^5}{5!}$ so $\frac{3^5}{5!} = \frac{81}{40}$.

33. $\left.\frac{d^3}{dx^3}(\ln x)\right|_{x=2} = \left.2x^{-3}\right|_{x=2} = \frac{1}{4}$

Therefore, the coefficient of $(x-2)^3$ is $\frac{\frac{1}{4}}{3!} = \frac{1}{24}$.

34. The linearization of f at a is the first order Taylor polynomial generated by f at $x = a$.

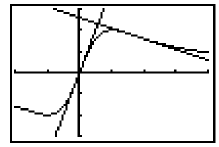
35. **(a)** Since $f'(x) = \frac{d}{dx}\left(\frac{4x}{x^2+1}\right)$

$$= \frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2}$$

$$= \frac{4-4x^2}{(x^2+1)^2},$$

we have $f(0) = 0$, $f'(0) = 4$, $f\left(\sqrt{3}\right) = \sqrt{3}$ and $f'\left(\sqrt{3}\right) = -\frac{1}{2}$, so the linearizations are $L_1(x) = 4x$ and

$L_2(x) = \sqrt{3} - \frac{1}{2}\left(x - \sqrt{3}\right) = -\frac{1}{2}x + \frac{3}{2}\sqrt{3}$, respectively.



[–2, 4] by [–3, 3]

(b) $f''(a)$ must be 0 because of the inflection point, so the second degree term in the Taylor series of f at $x = a$ is zero.

36. The series represents $\tan^{-1} x$. When $x = 1$, it converges to $\tan^{-1} 1 = \frac{\pi}{4}$. When $x = -1$, it converges to $\tan^{-1}(-1) = -\frac{\pi}{4}$.

37. True; since the series has no constant term, $f(0)$ must be 0.

38. False; the coefficient of x^3 is $-\frac{1}{3}$

$$\frac{f'''(0)}{3!} = -\frac{1}{3}$$

$$f'''(0) = -\frac{3!}{3} = -2.$$

39. E; $f(x) = 0 + x + 0 + \frac{2x^3}{3!}$

$$f(x) = \frac{1}{3}x^3 + x$$

40. A; $(-1)^2 \frac{3^4}{4!} = \frac{27}{8}$

41. C; $f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$

$$f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

$$f''\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$$

$$f'''\left(\frac{\pi}{2}\right) = -\cos\frac{\pi}{2} = 0$$

$$f^{(4)}\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$

$$P_4(x) = 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4$$

42. A; from the calculations above, we see that all the odd derivatives are zero. The second derivative is −1, which implies that the 6th, 10th, 14th . . . derivatives are also −1. The 4th derivative is 1, which implies that the 8th, 12th, 16th . . . derivatives are also 1.

Thus, the coefficient of $\left(x-\frac{\pi}{2}\right)^{2n}$ will be

$$\frac{f^{(2n)}\left(\frac{\pi}{2}\right)}{(2n)!}=\frac{(-1)^n}{(2n)!}.$$

The Taylor series is: $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\left(x-\frac{\pi}{2}\right)^{2n}}{(2n)!}.$

43. **(a)** $f(x)$

$$=\frac{1}{x}(\sin x)$$
$$=\frac{1}{x}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots\right)$$
$$=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots+(-1)^n\frac{x^{2n}}{(2n+1)!}+\cdots$$

(b) Because f is undefined at $x = 0$.

(c) $k = 1$

44. Note that the Maclaurin series for $\frac{1}{1-x}$ is $1+x+x^2+\cdots+x^n+\cdots$. If we differentiate this series and multiply by x, we obtain the desired Maclaurin series $x+2x^2+3x^3+\cdots+nx^n+\cdots$. Therefore, the desired function is

$$f(x)=x\frac{d}{dx}\left(\frac{1}{1-x}\right)=x\frac{1}{(1-x)^2}=\frac{x}{(x-1)^2}.$$

45. **(a)** $f(x)=(1+x)^m$

$$f'(x)=m(1+x)^{m-1}$$
$$f''(x)=m(m-1)(1+x)^{m-2}$$
$$f'''(x)=m(m-1)(m-2)(1+x)^{m-3}$$

(b) Differentiating $f(x)$ k times gives

$$f^{(k)}(x)$$
$$=m(m-1)(m-2)\cdots(m-k+1)(1+x)^{m-k}.$$

Substituting 0 for x, we have

$$f^{(k)}(0)=m(m-1)(m-2)\cdots(m-k+1).$$

(c) The coefficient is

$$\frac{f^{(k)}(0)}{k!}=\frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$$

(d) $f(0)=1,\ f'(0)=m,$ and we're done by part (c).

46. Because $f(x)=(1+x)^m$ is a polynomial of degree m. Alternately, observe that $f^{(k)}(0)=0$ for $k \ge m+1$.

Section 10.3 Taylor's Theorem (pp. 499–506)

Exploration 1 Your Turn

We need to consider what happens to $R_n(x)$ as $n\to\infty$. By Taylor's Theorem,

$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^n$, where $f^{(n+1)}(c)$ is the $(n+1)$st derivative of $\cos x$ evaluated at some c between x and 0. Depending on n, the $(n+1)$st derivative of $\cos x$ is either $\cos x$, $-\sin x$, $-\cos x$, or $\sin x$. Thus for all n and for all c, $-1\le f^{(n+1)}(c)\le 1$. Therefore, no matter what x is, we have

$$|R_n(x)|=\left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^n\right|$$
$$\le\left|\frac{1}{(n+1)!}x^n\right|$$
$$=\frac{|x|^n}{(n+1)!}.$$

As noted in Example 3, the factorial growth in the denominator eventually outstrips the power growth in the numerator, and we have $\frac{|x|^n}{(n+1)!}\to 0$ for all x. This means that $R_n(x)\to 0$ for all x, which completes the proof.